Growth

Authors : Bill Davis, Horacio Porta and Jerry Uhl
Producer : Bruce Carpenter
Publisher : Math Everywhere, Inc. **Distributor** : LiveMath LLC

1.03 Instantaneous Growth Rates

BASICS B1

B.1) Instantaneous growth rates

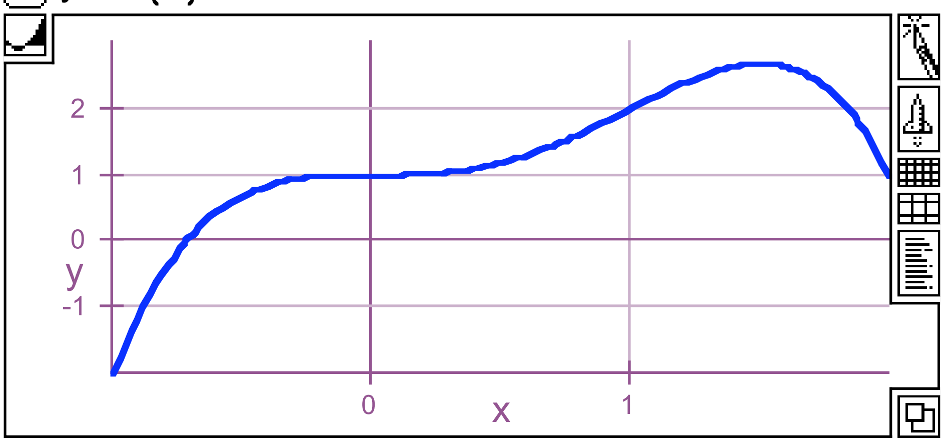
Here is a friendly function and a plot:

$f(x) = 1 + 2x^3 - x^4$

and a plot:

$f(x) = 1 + 2x^3 - x^4$

$y = f(x)$



B.1.a.i)

Measure the net growth of

$f(x) = 1 + 2x^3 - x^4$

over the interval [-1, 2].

Then measure the average growth rate of f(x) over the interval [-1, 2].

Answer:

Here you go:
Over the interval [-1,2], the function starts out at:

$f(x) = 1 + 2x^3 - x^4$

$f(-1)$

$f(-1) = -2$

And it ends up at:

$f(x) = 1 + 2x^3 - x^4$

$f(2)$

$f(2) = 1$

Its net growth is:

$f(x) = 1 + 2x^3 - x^4$

$fgrowth = f(2) - f(-1)$

$fgrowth = 3$

Its average growth rate in units on the y-axis per unit on the x-axis over the interval [-1, 2] is:

$f(x) = 1 + 2x^3 - x^4$

$fgrowth = f(2) - f(-1)$

$fgrowth = 3$

$xgrowth = 2 - (-1)$

$xgrowth = 3$

$$\frac{fgrowth}{xgrowth}$$

$$\frac{fgrowth}{xgrowth} = 1$$

As x grows from -1 to 2, on the average, f(x) grows at a rate of 1 unit every time x grows by 1 unit.
The average growth rate of f(x) over the interval [-1, 2] is 1.

B.1.a.ii)

Measure the average growth rate of

$f(x) = 1 + 2x^3 - x^4$

over the interval [x, x + 0.5].
Interpret the result.

Answer:

$f(x) = 1 + 2x^3 - x^4$

Over the interval [x, x + 0.5], the function starts out at:

$f(x) = 1 + 2x^3 - x^4$

$f(x)$

$f(x) = -x^4 + 2x^3 + 1$

And it ends up at:

$f(x) = 1 + 2x^3 - x^4$

$f(x + 0.5)$

$f(x + 0.5) = -(x + 0.5)^4 + 2(x + 0.5)^3 + 1$

Its net growth over the interval [x, x + 0.5] is:

$f(x) = 1 + 2x^3 - x^4$

$fgrowth = f(x + 0.5) - f(x)$

$fgrowth = -(x + 0.5)^4 + 2(x + 0.5)^3 + x^4 - 2x^3$

$fgrowth = -2x^3 + 1.5x^2 + x + 0.1875$

Its average growth rate in units on the y-axis per unit on the x-axis over the interval [x, x + 0.5] is:

$xgrowth = 0.5$

$f(x) = 1 + 2x^3 - x^4$

$fgrowth = f(x + 0.5) - f(x)$

$fgrowth = -(x + 0.5)^4 + 2(x + 0.5)^3 + x^4 - 2x^3$

$fgrowth = -2x^3 + 1.5x^2 + x + 0.1875$

$fAverageGrowthRate = \frac{fgrowth}{xgrowth}$

$fAverageGrowthRate = 2(-2x^3 + 1.5x^2 + x + 0.1875)$

$fAverageGrowthRate = -4x^3 + 3x^2 + 2x + 0.375$

On the interval [x, x + 0.5], the average growth rate of f(x) is

$0.375 + 2x + 3x^2 - 4x^3$.

For instance, when you look at:

$fAverageGrowthRate(x) = -4x^3 + 3x^2 + 2x + 0.375$

$fAverageGrowthRate(0)$

$fAverageGrowthRate(0) = 0.375$

then you see that, on the average, f(x) grows 0.375 times as fast as x grows as x advances from 0 to 0 + 0.5 = 0.5.

But when you look at:

$fAverageGrowthRate(x) = -4x^3 + 3x^2 + 2x + 0.375$

$fAverageGrowthRate(1.5)$

$fAverageGrowthRate(1.5) = -3.375$

Then you see that, on the average, f(x) goes down 3.375 times as fast as x grows as x advances from 1.5 to 1.5 + 0.5 = 2.

B.1.a.iii)

Given a positive number h, measure the average growth rate of

$f(x) = 1 + 2x^3 - x^4$

over the interval [x, x + h].

Interpret the result.

Answer:

$f(x) = 1 + 2x^3 - x^4$

Over the interval [x, x + h], the function starts out at:

$f(x) = 1 + 2x^3 - x^4$

$f(x)$

$f(x) = -x^4 + 2x^3 + 1$

And it ends up at:

$f(x) = 1 + 2x^3 - x^4$

$f(x + h)$

$f(x + h) = -(h + x)^4 + 2(h + x)^3 + 1$

$f(x + h) = -h^4 + 2h^3 - x^4 - 4hx^3 + 2x^3 - 6h^2x^2 + 6hx^2 - 4h^3x + 6h^2x + 1$

Its net growth over the interval [x, x + h] is:

$f(x) = 1 + 2x^3 - x^4$

$fgrowth = f(x + h) - f(x)$

$fgrowth = -(h + x)^4 + 2(h + x)^3 + x^4 - 2x^3$

$fgrowth = -h^4 + 2h^3 - 4hx^3 - 6h^2x^2 + 6hx^2 - 4h^3x + 6h^2x$

Its average growth rate in units on the y-axis per unit on the x-axis over the interval [x, x + h] is:

$f(x) = 1 + 2x^3 - x^4$

$\text{xgrowth} = h$

$\text{fAverageGrowthRate}(x, w) = \dfrac{f(x + w) - f(x)}{w}$

$\text{fAverageGrowthRate}(x, w) = \dfrac{-(w + x)^4 + 2(w + x)^3 + x^4 - 2x^3}{w}$

$\text{fAverageGrowthRate}(x, w) = \dfrac{-6(w\,x)^2 - w^4 + 2w^3 - 4w\,x^3 + 6w\,x^2 - \ldots}{w}$

$\text{fAverageGrowthRate}(x, w) = -6\dfrac{1}{w}(w\,x)^2 - w^3 + 2w^2 - 4x^3 + 6x^2 - \ldots$

$\text{fAverageGrowthRate}(x, w) = -6w\,x^2 - w^3 + 2w^2 - 4x^3 + 6x^2 - 4w^2x + 6w\ldots$

$\text{fAverageGrowthRate}(x, h)$

$\text{fAverageGrowthRate}(x, h) = -h^3 + 2h^2 - 4x^3 - 6h\,x^2 + 6x^2 - 4h^2x + 6h\,x$

This gives you the measurement of the average growth rate of f(x) on the interval [x, x + h]. For instance when you look at:

$\text{fAverageGrowthRate}(x, w) = -6w\,x^2 - w^3 + 2w^2 - 4x^3 + 6x^2 - 4w^2x + 6w\ldots$

$\text{fAverageGrowthRate}(0, h)$

$\text{fAverageGrowthRate}(0, h) = -h^3 + 2h^2$

Then you see that, on the average, f(x) grows

$2h^2 - h^3$

times as fast as x grows as x advances from 0 to 0 + h.
But when you look at:

$\text{fAverageGrowthRate}(x, w) = -6w\,x^2 - w^3 + 2w^2 - 4x^3 + 6x^2 - 4w^2x + 6w\ldots$

$\text{fAverageGrowthRate}(1, h)$

$\text{fAverageGrowthRate}(1, h) = -h^3 - 2h^2 + 2$

Then you see that, on the average, f grows

$2 - 2h^2 - h^3$

times as fast as x grows as x advances from 1 to 1 + h.

B.1.a.iv) The instantaneous growth rate

Measure the instantanteous growth rate f'(x) of

$f(x) = 1 + 2x^3 - x^4$

at a point x.

Answer:

$f(x) = 1 + 2x^3 - x^4$

Its average growth rate in units on the y-axis per unit on the x-axis over the interval [x, x + h] is:

$f(x) = 1 + 2x^3 - x^4$

xgrowth = h

$\text{fAverageGrowthRate}(x, w) = \dfrac{f(x + w) - f(x)}{w}$

$\text{fAverageGrowthRate}(x, w) = \dfrac{-(w + x)^4 + 2(w + x)^3 + x^4 - 2x^3}{w}$

$\text{fAverageGrowthRate}(x, w) = \dfrac{-6(w\,x)^2 - w^4 + 2w^3 - 4w\,x^3 + 6w\,x^2 - \ldots}{w}$

$\text{fAverageGrowthRate}(x, w) = -6\dfrac{1}{w}(w\,x)^2 - w^3 + 2w^2 - 4x^3 + 6x^2 - \ldots$

$\text{fAverageGrowthRate}(x, w) = -6(w\,x^2) - w^3 + 2w^2 - 4x^3 + 6x^2 - 4w^2x + 6\ldots$

fAverageGrowthRate(x, h)

$\text{fAverageGrowthRate}(x, h) = -h^3 + 2h^2 - 4x^3 - 6hx^2 + 6x^2 - 4h^2x + 6hx$

The instantaneous rate of change at x is the limiting case of the above average rate as h closes in on 0.

Evidently this is

$$0 - 0 + 0 - 0 + 6x^2 - 0 - 4x^3 = 6x^2 - 4x^3.$$

So the instantaneous growth rate of $f(x) = 1 + 2x^3 - x^4$ at a point x in units on the y-axis per unit on the x-axis is given by

$$f'(x) = 6x^2 - 4x^3.$$

LiveMath knows how to calculate the instantaneous growth rate f'(x). You just have to ask for it.

$$f(x) = 1 + 2x^3 - x^4$$

$$\frac{\partial}{\partial x} f(x)$$

$$\frac{\partial}{\partial x} f(x) = -4x^3 + 6x^2$$

Nice work, LiveMath.

For instance, when you look at:

$$f(x) = 1 + 2x^3 - x^4$$

$$\frac{\partial}{\partial x} f(x)$$

$$\frac{\partial}{\partial x} f(x) = -4x^3 + 6x^2$$

$$f'(x) = -4x^3 + 6x^2$$

$$f'(1.2)$$

$$f'(1.2) = 1.728$$

then you see that, as x advances through 1.2, f(x) grows

1.728 times

as fast as x grows.

This tells you that f(x) goes up as x advances through 1.2.

But when you look at:

$f(x) = 1 + 2x^3 - x^4$

$\frac{\partial}{\partial x} f(x)$

$\frac{\partial}{\partial x} f(x) = -4x^3 + 6x^2$

$f'(x) = -4x^3 + 6x^2$

$f'(1.6)$

$f'(1.6) = -1.024$

then you see that, as x advances through 1.6, f(x) grows

-1.024 times

as fast as x grows.

This tells you that f(x) goes down as x advances through 1.6.

Nice work, LiveMath.

B.1.b.i)

Here is a plot of

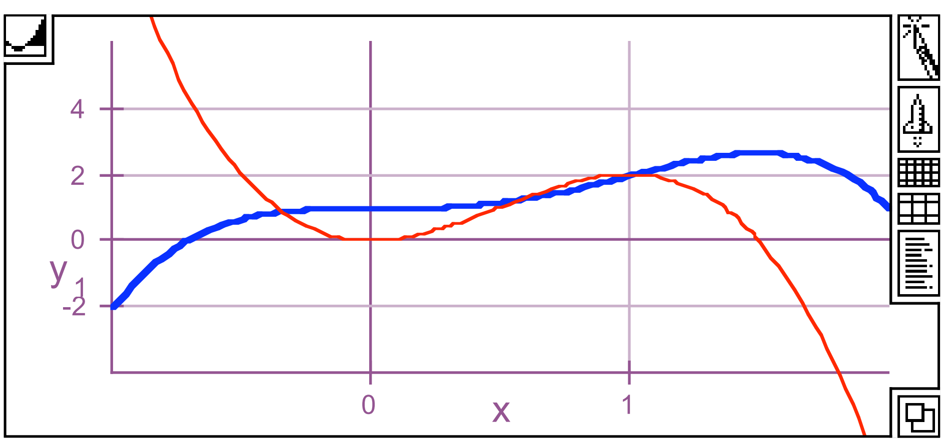
$f(x) = 1 + 2x^3 - x^4$

together with a plot of the function f'(x) that measures the instantaneous growth rate of f(x) at a point x.

$f(x) = 1 + 2x^3 - x^4$

$y_1 = f(x)$

$y_2 = \frac{\partial}{\partial x} f(x)$



The plot of the instantaneous growth rate f'(x) is the thinner of the two plots.

Interpret the relationship between the two plots.

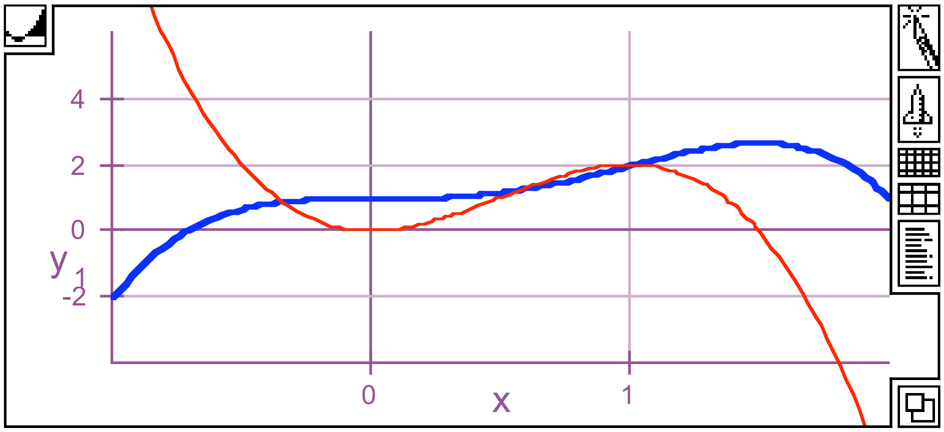
Answer:

Take another look:

$f(x) = 1 + 2x^3 - x^4$

$y_1 = f(x)$

$y_2 = \frac{\partial}{\partial x} f(x)$



The plot of the instantaneous growth rate f'(x) is the thinner of the two plots.

Here are the things that should grab your attention:
→ When f'(x) is positive, then f(x) is going up.
→ When f'(x) is negative, then f(x) is going down.
These are both natural because f'(x) measures the instantaneous growth rate of f(x) at x.

B.1.b.ii)

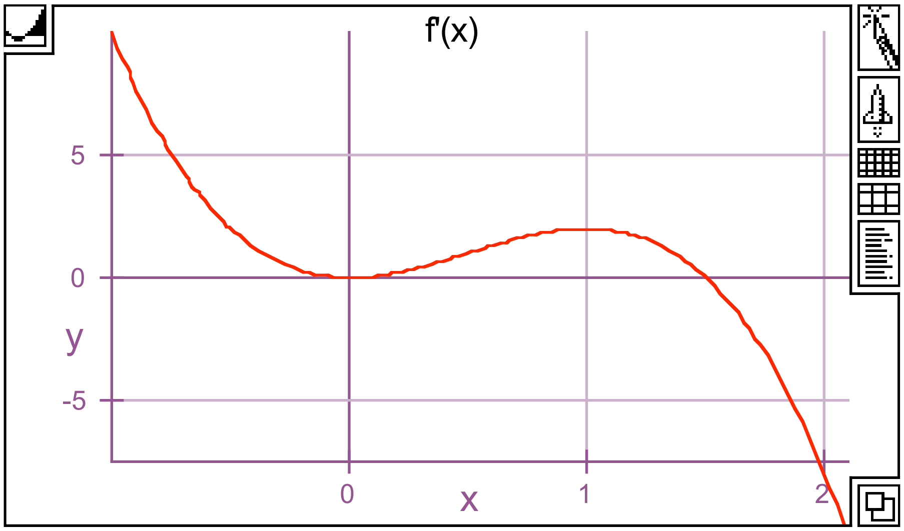
Here is a plot of the function f'(x) that measures the instantaneous growth rate of

$f(x) = 1 + 2x^3 - x^4$

at a point x.

$$f(x) = 1 + 2x^3 - x^4$$

$$y = \frac{\partial}{\partial x} f(x)$$



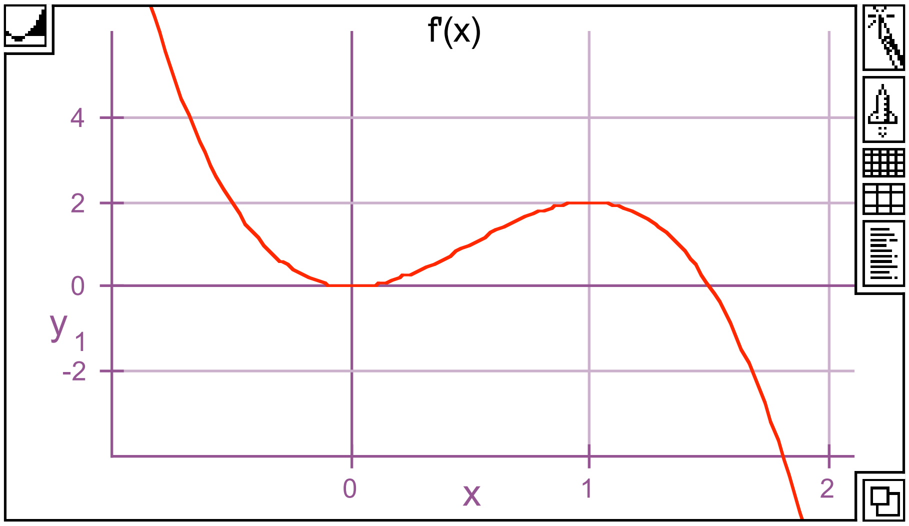
Use this plot of the instantaneous growth rate f'(x) to read off the answers to the following three questions:
At what point in [-1, 2] is f(x) going up most rapidly?
At what point in [-1, 2] is f(x) going down most rapidly?
At what points in [-1, 2] is f(x) changing least rapidly?

Answer:

Take another look at the plot of the instantaneous growth rate f'(x) for $-1 \leq x \leq 2$.

$f(x) = 1 + 2x^3 - x^4$

$y = \frac{\partial}{\partial x} f(x)$

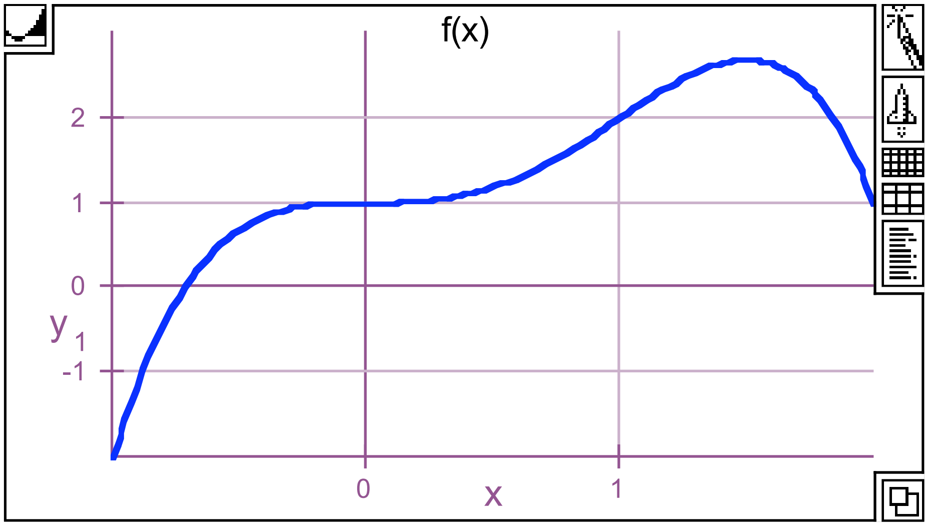


This is a plot of $f'(x)$ for $-1 \le x \le 2$.
This plot tells you that:
→ The instantaneous growth rate $f'(x)$ is highest at the left endpoint $x = -1$, so the function $f(x)$ is going up most rapidly at $x = -1$.
→ The instantaneous growth rate $f'(x)$ is lowest at the right endpoint $x = 2$, so the function $f(x)$ is going down most rapidly at $x = 2$.
→ The instantaneous growth rate $f'(x)$ is 0 at $x = 0$ and at $x = 1.5$, so the function $f(x)$ is changing the least rapidly at $x = 0$ and $x = 1.5$.

Take a look at the plot of a to make sure that this makes sense:

$f(x) = 1 + 2x^3 - x^4$

$y = f(x)$



Sure enough:
Rapid increase at the left endpoint, rapid decrease at the right endpoint, and lazy growth at $x = 0$ and $x = 1.5$.

B.1.c.i)

Here is a plot on [0, 12] of

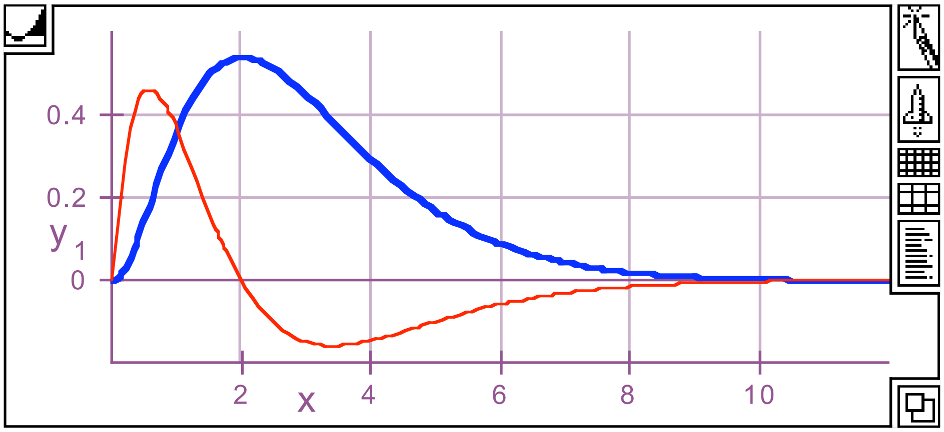
$f(x) = x^2 e^{-x}$

together with a plot of the the function f'(x) that measures the instantaneous growth rate of f(x) at a point x.

$f(x) = x^2 e^{-x}$

$y_1 = f(x)$

$y_2 = \frac{\partial}{\partial x} f(x)$



The plot of the instantaneous growth rate $f'(x)$ is the thinner of the two plots.

Interpret the relationship between the two plots.

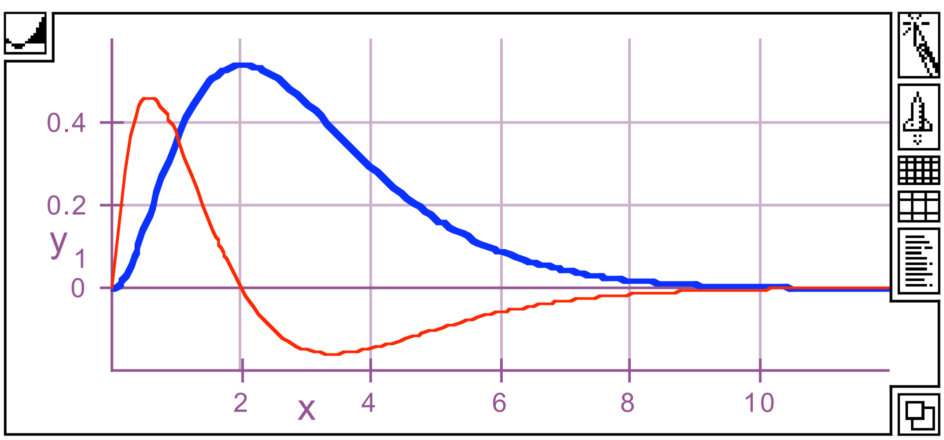
Answer:

Take another look:

$f(x) = x^2 e^{-x}$

$y_1 = f(x)$

$y_2 = \frac{\partial}{\partial x} f(x)$



The plot of the instantaneous growth rate $f'(x)$ is the thinner of the two plots.

Again:
→ When f'(x) is positive, then f(x) is going up.
→ When f'(x) is negative, then f(x) is going down.
These are both natural because f'(x) measures the instantaneous growth rate of f(x) at x.

B.1.c.ii)

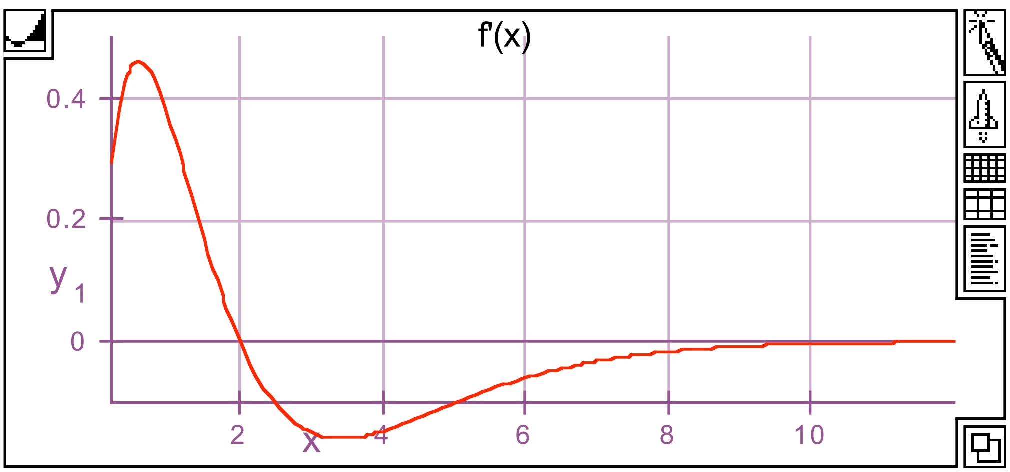
Here is a plot on [0.2, 12] of the function f'(x) that measures the instantaneous growth rate of

$f(x) = x^2 e^{-x}$

at a point x.

$f(x) = x^2 e^{-x}$

$y = \frac{\partial}{\partial x} f(x)$



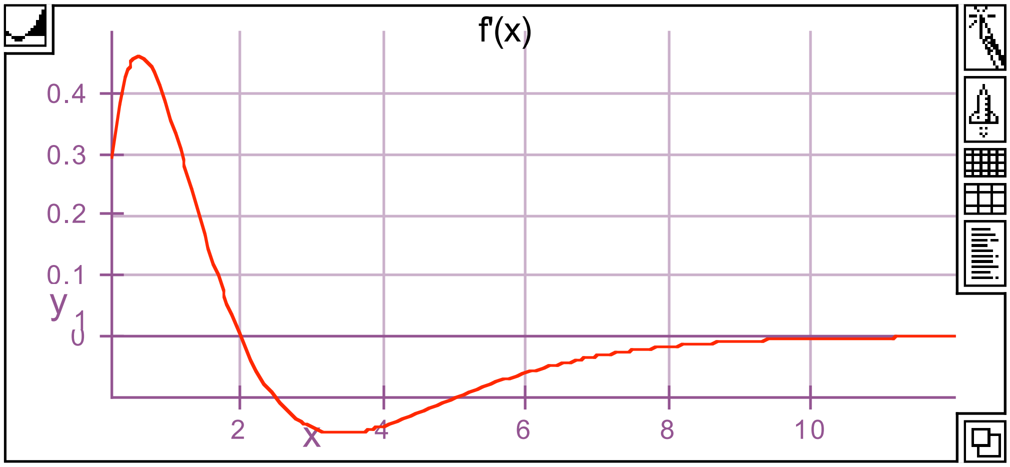
Use this plot of the instantaneous growth rate f'(x) to read off the answers to the following three questions:
At what point in [0.2, 12] is f(x) going up most rapidly?
At what point in [0.2, 12] is f(x) going down most rapidly?
At what points in [0.2, 12] is f(x) changing least rapidly?

Answer:

Take another look at the plot of the instantaneous growth rate f'(x).

$f(x) = x^2 e^{-x}$

$y = \frac{\partial}{\partial x} f(x)$



This is a plot of $f'(x)$ for $0.2 \le x \le 12$.

This plot tells you that:

→ The instantaneous growth rate $f'(x)$ is highest near the point $x = 0.6$, so the function $f(x)$ is going up most rapidly near $x = 0.6$.

→ The instantaneous growth rate $f'(x)$ is lowest near the point $x = 3.4$, so the function $f(x)$ is going down most rapidly near $x = 3.4$.

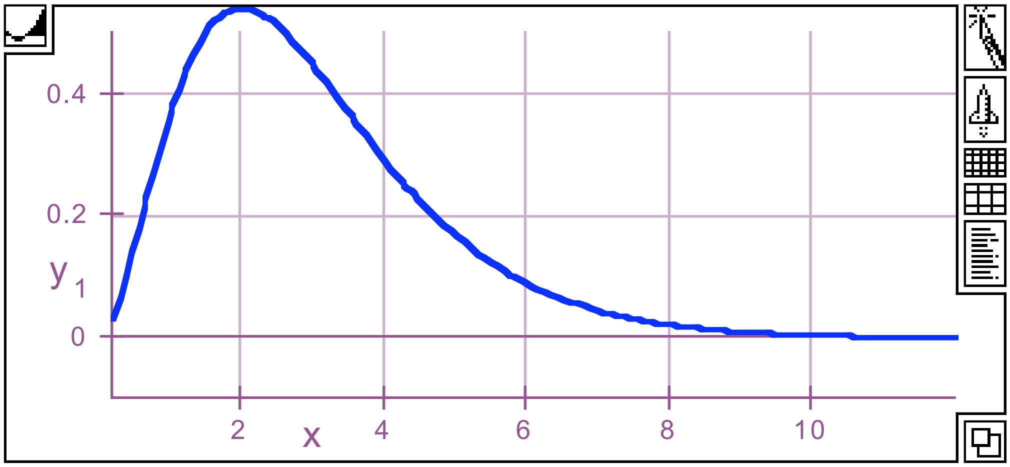
→ The instantaneous growth rate $f'(x)$ is 0 at $x = 2$, so the function $f(x)$ is changing the least rapidly at $x = 2$.

→ The instantaneous growth rate $f'(x)$ is near 0 for $10 \le x \le 12$ so the function is changing very little over the interval $10 \le x \le 12$.

Take a look at the plot of $f(x)$ to make sure that t his makes sense.

$f(x) = x^2 e^{-x}$

$y = f(x)$



Sure enough:
Rapid increase near $x = 0.6$, rapid decrease near $x = 3.4$, and lazy growth at $x = 2$ and at x's within the interval $10 \leq x \leq 12$.