Growth

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1.07 The Race Track Principle

Give It a Try G10

Graphics Primitives

DESolvers

This LiveMath Independence Declaration allows the derivatives to be computed as intended in this notebook.

The variables $(a, b, c, x, y, t, r, k, s, z)$ are independent of each other

LiveMath Note: y vs. Y

G.10) The Law of the Mean

G.10.a.i)

Here is a function f(x) and a plot of its derivative on the interval $a \le x \le b$ for $a = 1$ and $b = 2$:

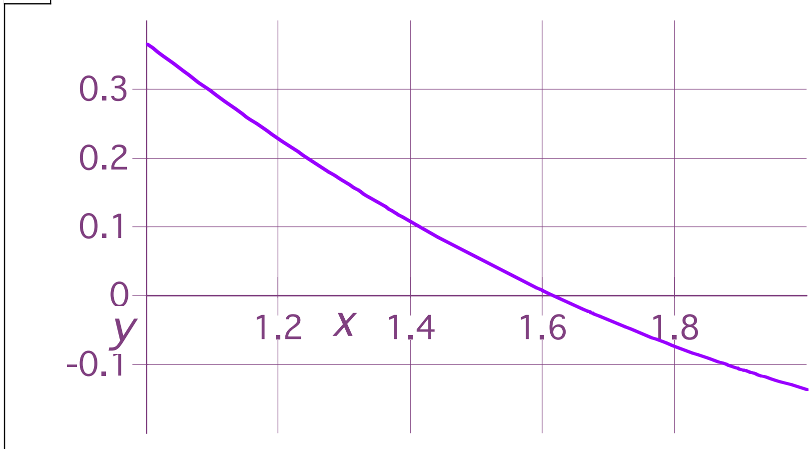
$a = 1$

$b = 2$

$f(\mathbb{x}) = (\mathbb{x}^2 + \mathbb{x})e^{-\mathbb{x}}$

$f'(\mathbb{x}) = \left[\frac{d}{dx} f(x)\right]_{x = \mathbb{x}}$

$f'(\mathbb{x}) = -(\mathbb{x}^2 + \mathbb{x})e^{-\mathbb{x}} + (2\mathbb{x} + 1)e^{-\mathbb{x}}$ *Substitute*



$a \ldots b$ = left...right Stretch to Fit

$-0.2 \ldots 0.4$ = bottom...top cropping

Moderately Wide

Graph Building Blocks

Curve at $(x, f'[x])$ where x = left ... right

with a heavy line, colored Purple.

Read from the plot that

$$\text{mingrowth} = f'(2) \le f'(x) \le f'(1) = \text{maxgrowth}$$

for all the x's with $a \le x \le b$.

Now plot on the same interval f(x) and the lines that go through {a

with constant growth rates mingrowth and maxgrowth:

$a = 1$

$b = 2$

$f(x) = (x^2 + x)e^{-x}$

$f'(x) = \left. \frac{d}{dx} f(x) \right|_{x = x}$

$f'(x) = -(x^2 + x)e^{-x} + (2x + 1)e^{-x}$ *Substitute*

$\text{mingrowth} = f'(2)$

$\text{maxgrowth} = f'(1)$

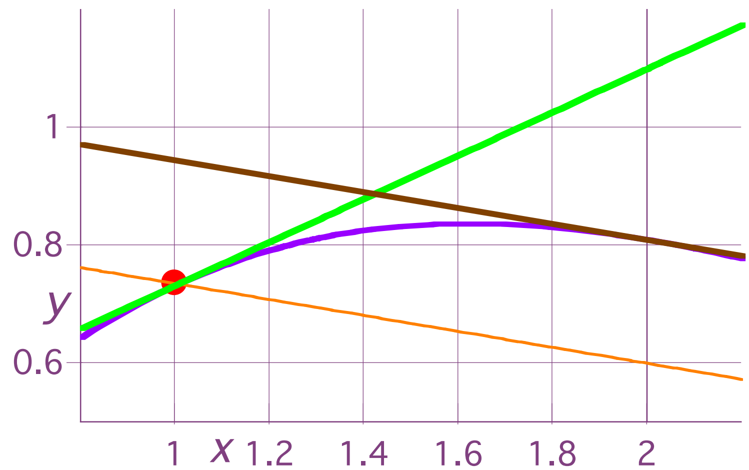
$\text{upline}(x) = \text{maxgrowth}(x - a) + f(a)$

$\text{downline}(x) = \text{mingrowth}(x - a) + f(a)$

RC: 03/17/12:

f '(2) is the slope of the Brown line = tangent at x=2

f '(1) is the slope of the Green line = tangent at x=1



$a - 0.2 \ldots b + 0.2$ = left...right

Stretch to Fit

0.5 ... 1.2 = bottom...top cropping

Moderately Wide

Graph Building Blocks

Curve at $(x, f[x])$ where x = left ... right with a extra heavy line, colored Purple.

Scatter plot of $(a, f[a])$ where ? using 10 point spots colored Red.

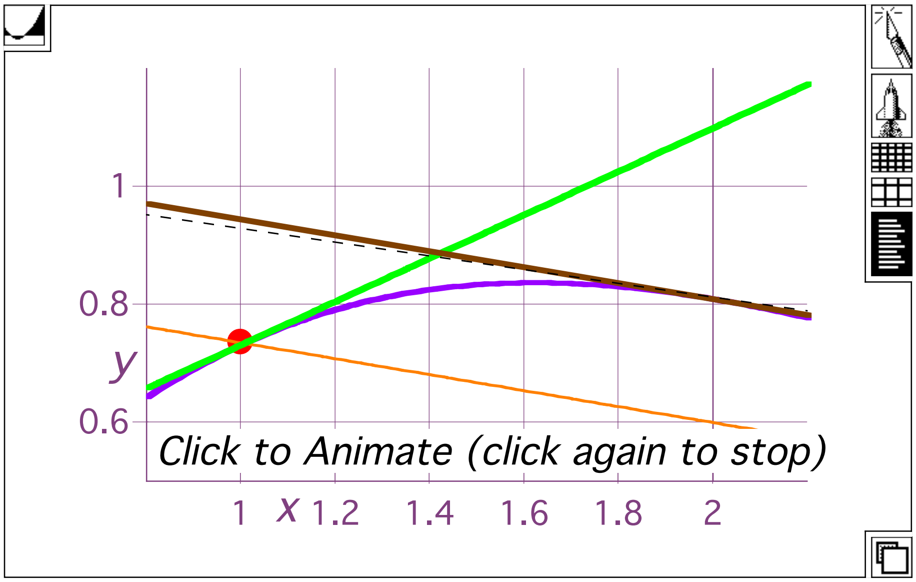
Curve at $(x, \text{downline}[x])$ where x = left ... right with a heavy line, colored Orange.

Curve at $(x, \text{upline}[x])$ where x = left ... right with a extra heavy line, colored Green.

Curve at $(x, \text{mingrowth}[x - b] + f[b])$

where x = left ... right with a extra heavy line, colored Brown.

RC: 03/17/12:
The statement you are trying to establish is:
f '(1) ≤ f '(x) ≤ f '(2)
for all x in [1,2]. This inequality is demonstrated in this animation:



Animate this graph for $c = 1 \ldots 2$ in steps of $\frac{1}{30}$ for a total of 30 frames in a cycle at 6 frames/second.

$a - 0.2 \ldots b + 0.2$ = left...right
Stretch to Fit
$0.5 \ldots 1.2$ = bottom...top cropping
Moderately Wide

Graph Building Blocks

Curve at $(x, f[x])$ where x = left ... right with a extra heavy line, colored Purple.

Scatter plot of $(a, f[a])$ where ? using 10 point spots colored Red.

Curve at $(x, \text{downline}[x])$ where x = left ... right with a heavy line, colored Orange.

M7.3.j-2=3.the

Curve at $(x, \text{upline}[x])$ where $x = \text{left} \ldots \text{right}$ with a extra heavy line, colored Green.

Curve at $(x, \text{mingrowth}[x - b] + f[b])$ where $x = \text{left} \ldots \text{right}$ with a extra heavy line, colored Brown.

Curve at $(x, f'[c][x - c] + f[c])$ where $x = \text{left} \ldots \text{right}$ with a dashed line, colored Black.

RC: 03/17/12: Here is an example of a function that does not have this property of

f '(1) ≤ f '(x) ≤ f '(2)

Give a description from the animation about the observation that this inequality fails.

$a = 1$

$b = 2$

$f(x) = \dfrac{1}{x - 1.5}$

$f'(x) = \left. \dfrac{d}{dx} f(x) \right|_{x = x}$

$f'(x) = -\dfrac{1}{(x - 1.5)^2}$ *Substitute*

$\text{mingrowth} = f'(2)$

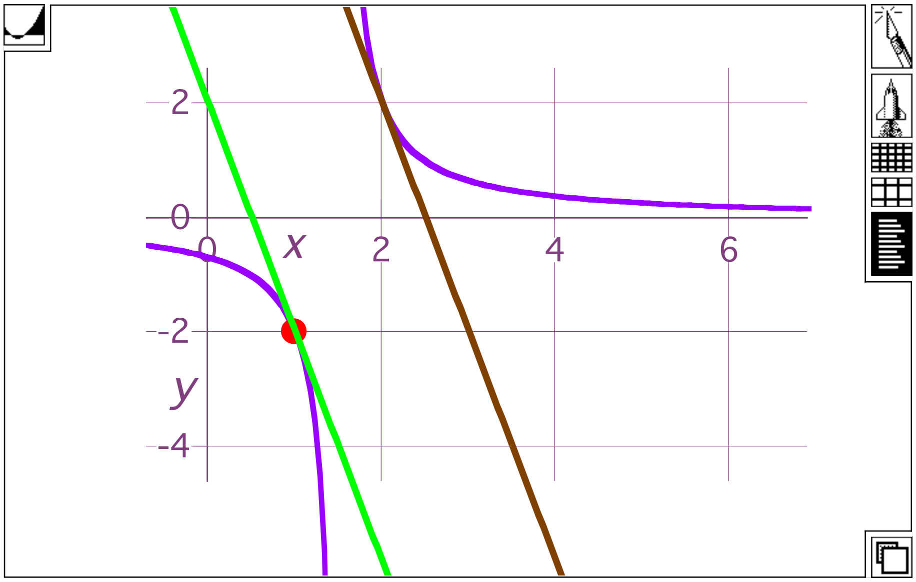
$\text{maxgrowth} = f'(1)$

$\text{upline}(x) = \text{maxgrowth}(x - a) + f(a)$

$\text{downline}(x) = \text{mingrowth}(x - a) + f(a)$

RC: 03/17/12:

f '(2) is the slope of the Brown line = tangent at x=2 f '(1) is the slope of the Green line = tangent at x=1



$\frac{17}{2}a-\frac{15}{2}b+5.8 \ldots \frac{17}{2}a-\frac{15}{2}b+13.4$ = left...right Stretch to Fit

$-4.6 \ldots 2.6$ = bottom...top cropping Moderately Wide

Graph Building Blocks

Curve at $(x, f[x])$ where x = left ... right with a extra heavy line, colored Purple.

Scatter plot of $(a, f[a])$ where ? using 10 point spots colored Red.

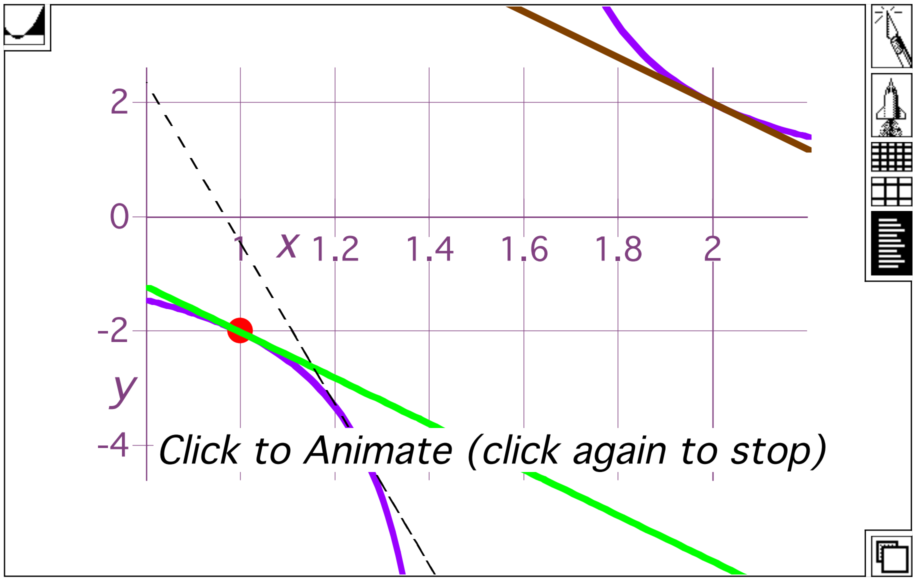
Curve at $(x, \text{downline}[x])$ where x = left ... right with a heavy line, colored Orange.

Curve at $(x, \text{upline}[x])$ where x = left ... right with a extra heavy line, colored Green.

Curve at $(x, \text{mingrowth}[x-b]+f[b])$ where x = left ... right with a extra heavy line, colored Brown.

RC: 03/17/12:

The statement you are trying to establish is:
f '(1) ≤ f '(x) ≤ f '(2)
for all x in [1,2]. This inequality is demonstrated in this animation:



Animate this graph for $c = 1 \ldots 2$ in steps of $\frac{1}{30}$ for a total of 30 frames [in a cycle ▾] at [6 frames/second ▾].

$a - 0.2 \ldots b + 0.2$ = left...right
[Stretch to Fit ▾]
$-4.6 \ldots 2.6$ = bottom...top cropping
[Moderately Wide ▾]

Graph Building Blocks

Curve at $(x, f[x])$ where x = left ... right with a [extra heavy ▾] line, colored [Purple ▾].

Scatter plot of $(a, f[a])$ where ? using 10 point [spots ▾] colored [Red ▾].

Curve at $(x, \text{downline}[x])$ where x = left ... right with a [heavy ▾] line, colored [Orange ▾].

Curve at $(x, \text{upline}[x])$ where x = left ... right with a [extra heavy ▾] line, colored [Green ▾].

Curve at $(x, \text{mingrowth}[x - b] + f[b])$

where x = left ... right with a extra heavy line, colored Brown.

Curve at $(x, f'[c][x - c] + f[c])$ where x = left ... right with a dashed line, colored Black.

RC: 03/17/12: Now the question: can we monkey around with a function that does not have an asymptote like this previous example to get this inequality

f '(1) ≤ f '(x) ≤ f '(2)

to fail? Try some examples and see if you can make this inequality fail with a " nice function". I'll start you out

$a = 1$

$b = 2$

$f(x) = \sin(5x)$

$f'(x) = \left. \frac{d}{dx} f(x) \right|_{x = x}$

$f'(x) = 5\cos(5x)$ *Substitute*

mingrowth $= f'(2)$

maxgrowth $= f'(1)$

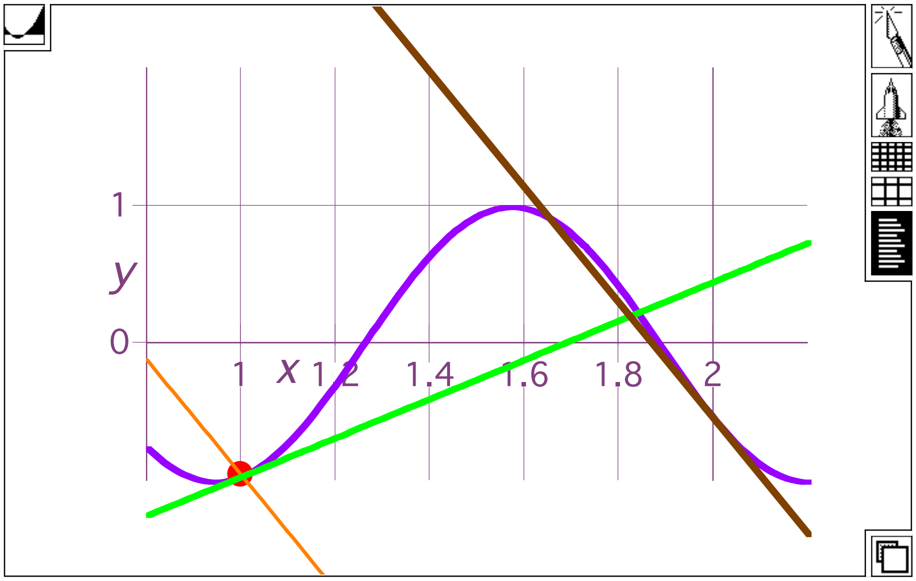
upline(x) = maxgrowth$(x - a) + f(a)$

downline(x) = mingrowth$(x - a) + f(a)$

RC: 03/17/12:

f '(2) is the slope of the Brown line = tangent at x=2

f '(1) is the slope of the Green line = tangent at x=1



a – 0.2 ... b + 0.2 = left...right

Stretch to Fit

– 1 ... 2 = bottom...top   cropping

Moderately Wide

Graph Building Blocks

Curve at (x, f[x]) where x = left ... right with a extra heavy line, colored Purple.

Scatter plot of (a, f[a]) where ? using 10 point spots colored Red.

Curve at (x, downline[x]) where x = left ... right with a heavy line, colored Orange.

Curve at (x, upline[x]) where x = left ... right with a extra heavy line, colored Green.

Curve at (x, mingrowth[x – b] + f[b]) where x = left ... right with a extra heavy line, colored Brown.

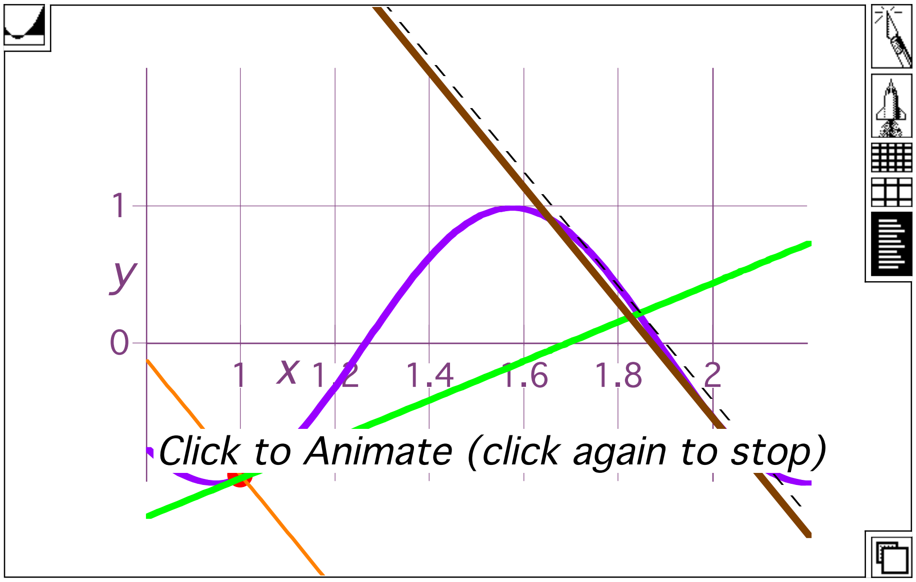
RC: 03/17/12:

The statement you are trying to establish is:

$f'(1) \le f'(x) \le f'(2)$

for all x in [1,2]. This inequality is demonstrated in this animation.

Maybe here mingrowth and maxgrowth are interchanged.



Animate this graph for $c = 1 \ldots 2$ in steps of $\frac{1}{30}$ for a total of 30 frames [in a cycle ▼] at [6 frames/second ▼].

$a - 0.2 \ldots b + 0.2$ = left...right [Stretch to Fit ▼]

$-1 \ldots 2$ = bottom...top cropping [Moderately Wide ▼]

Graph Building Blocks

Curve at $(x, f[x])$ where x = left ... right with a [extra heavy ▼] line, colored [Purple ▼].

Scatter plot of $(a, f[a])$ where ? using 10 point [spots ▼] colored [Red ▼].

Curve at $(x, \text{downline}[x])$ where x = left ... right with a [heavy ▼] line, colored [Orange ▼].

Curve at $(x, \text{upline}[x])$ where x = left ... right with a [extra heavy ▼] line, colored [Green ▼].

Curve at $(x, \text{mingrowth}[x - b] + f[b])$ where x = left ... right with a [extra heavy ▼] line, colored [Brown ▼].

Curve at $(x, f'[c][x-c]+f[c])$ where x = left ... right with a dashed line, colored Black.

Why did the plot turn out this way?

MM: 2/19/12: Because on the left side of the graph the maximum growth is positive and on the right side of the graph the minimum growth is negative.

AP 3/5/12: Not quite -- look at upline and downline -- where do they get their slopes from? How does this relate to the slope of the tangent at this given point (not pictured here)? No matter what point you look at here, the function will always graph in between downline and upline. Why?

MM: 3/8/12: The upline represents the max growth for the given point and the downline represents the min growth. The average of the upline and the downline growth equals the slope of the tangent line. Since the tangent line equals the instantaneous slope of the plot at the given point, the plot will always be in between the upline and downline.

RC: 03/17/12: Nope. See instructions above.

G.10.a.ii)

Here is another function f(x) and a plot of its derivative on the inte

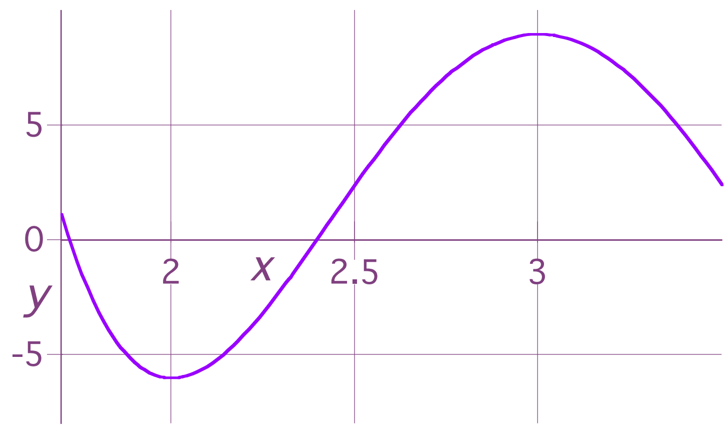
$a \le x \le b$ for $a = 1.7$ and $b = 3.5$:

$a = 1.7$
$b = 3.5$

$f(\mathbb{x}) = -447 + 954\mathbb{x} - 720\mathbb{x}^2 + 260\mathbb{x}^3 - 45\mathbb{x}^4 + 3\mathbb{x}^5$

$f'(\mathbb{x}) = \left[\frac{d}{dx} f(x)\right]_{x=\mathbb{x}}$

$f'(\mathbb{x}) = 15\mathbb{x}^4 - 180\mathbb{x}^3 + 780\mathbb{x}^2 - 1440\mathbb{x} + 954$ *Substitute*



a ... *b* = left...right Stretch to Fit

– 8 ... 10 = bottom...top cropping

Moderately Wide

Graph Building Blocks

Curve at $(x, f'[x])$ where x = left ... right

with a heavy line, colored Purple.

Read from the plot that

mingrowth = f ′(2) ≤ f ′(x) ≤ f ′(3) = maxgrowth

for all the x's with a ≤ x ≤ b where a = 1.7 and b = 3.5.

Now plot on the same interval f(x) and the lines that go through {a

with constant growth rates mingrowth and maxgrowth:

$a = 1.7$

$b = 3.5$

$f(\mathbb{x}) = -447 + 954\mathbb{x} - 720\mathbb{x}^2 + 260\mathbb{x}^3 - 45\mathbb{x}^4 + 3\mathbb{x}^5$

$f'(\mathbb{x}) = \left[\frac{d}{dx} f(x)\right]_{x=\mathbb{x}}$

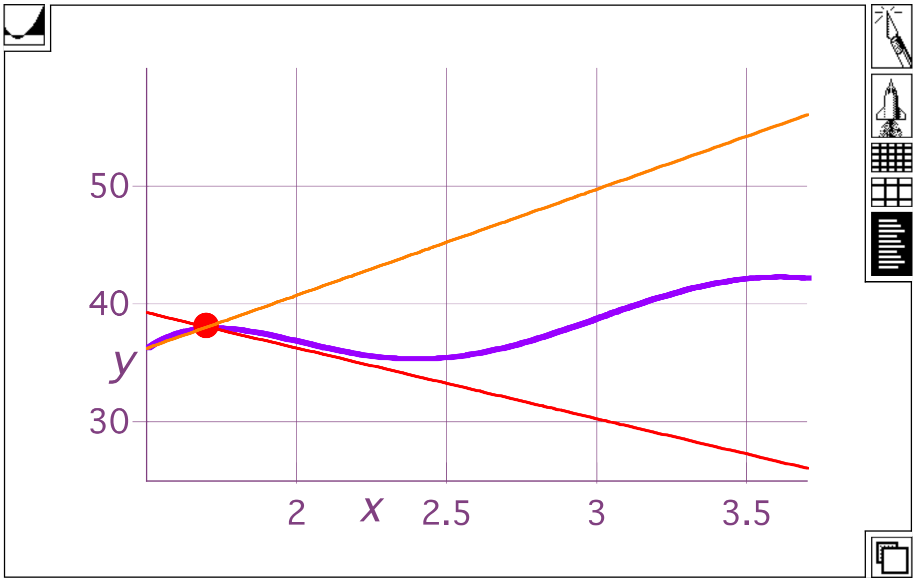
$f'(x) = 15x^4 - 180x^3 + 780x^2 - 1440x + 954$ *Substitute*

mingrowth $= f'(2)$

maxgrowth $= f'(3)$

upline$(x) =$ maxgrowth$(x - a) + f(a)$

downline$(x) =$ mingrowth$(x - a) + f(a)$



$a - 0.2 \ldots b + 0.2$ = left...right

Stretch to Fit

25 ... 60 = bottom...top cropping

Moderately Wide

Graph Building Blocks

Curve at $(x, f[x])$ where x = left ... right with a extra heavy line, colored Purple.

Scatter plot of $(a, f[a])$ where ? using 10 point spots colored Red.

Curve at $(x,$ downline$[x])$ where x = left ... right with a heavy line, colored Red.

Curve at $(x,$ upline$[x])$ where x = left ... right with a heavy line, colored Orange.

Why did the plot turn out this way?

MM: 2/19/12: Because the maximum growth is positive and the minimum growth is negative.

AP 3/5/12: see note on last problem

MM: 3/8/12: The upline represents the max growth for the given point and the downline represents the min growth. The average of the upline and the downline growth equals the slope of the tangent line. Since the tangent line equals the instantaneous slope of the plot at the given point, the plot will always be in between the upline and downline.

G.10.b.i)

Given $f(x)$, a and b with $a < b$.

If you plot $f'(x)$ for $a \le x \le b$ and read off, as above, the

$$\text{maximum growth rate} = \text{maxgrowth}$$

and the

$$\text{minimum growth rate} = \text{mingrowth}$$

and then you put

$$\text{upline}(x) = \text{maxgrowth}\,(x - a) + f(a)$$

and

$$\text{downline}(x) = \text{mingrowth}\,(x - a) + f(a),$$

why are you sure that

$$\text{downline}(b) \le f(b) \le \text{upline}(b)?$$

Tip:

Race Track Principle.

MM: 2/19/12: Using the Race track principle we know that the upline(x) of any value will always beat out both of the others, and f(x) will always beat out

M7.3.j-2=3.the

MM: 2/19/12: Using the Race track principle we know that the upline(x) of any value will always beat out both of the others, and f(x) will always beat out downline(x) simply because downline(x) represents the minimum and upline(x) represents the maximum.

G.10.b.ii)

The inequality

$$\text{downline}(b) \le f(b) \le \text{upline}(b)$$

above is the same as

$$\text{mingrowth}\,(b - a) + f(a)$$
$$\le f(b)$$
$$\le \text{maxgrowth}\,(b - a) + f(a).$$

This is the same as

$$\text{mingrowth}\,(b - a)$$
$$\le f(b) - f(a)$$
$$\le \text{maxgrowth}\,(b - a).$$

Explain why this tells you that you can expect there to be a disting

$$a \le c \le b$$

such that

$$f(b) - f(a) = f'(c)\,(b - a).$$

This is often called the Law of the Mean.

G.10.b.iii)

The Law of the Mean, as above, says that if you are given a and b w

then you can expect there to be a happy camper c with $a \le c \le b$ s

$$f(b) - f(a) = f'(c)\,(b - a).$$

In traditional courses, the Law of the Mean is proved using a quick

and then versions of the Race Track Principle are sometimes allude

Just for the fun of it, try to use the Law of the Mean to explain why

$f'(x) > 0$ for $a \le x \le b$,

then

$f(b) > f(a)$.

Tip:

That

$f(b) > f(a)$

follows directly from the Race Track Principle or other considerati
The point of this problem is to see whether you can use the Law of the Mean (and nothing else) to deduce that

$f(b) > f(a)$.

This is the sort of stuff abstract mathematicians really get off or
It's even fun.

MM: 2/19/12: Im not sure where to start with this one.

AP 3/5/12: Start with b.ii and explain why there must be some value between the max and min that is a mean growth rate. Then here's a hint:

By the Law of the Mean, there exists c, with $a \le c \le b$, such that $f'(c)(b - a) = f(b) - f(a)$. And we're given that f ' is positive in the interval; so $f'(c) > 0$.

MM: 3/8/12: The upline represents the max growth for the given point and the downline represents the min growth. The average of the upline and the downline growth equals the slope of the tangent line. Since the tangent line equals the instantaneous slope of the plot at the given point, the plot will always be in between the upline and downline. In the definition of the Law of the Mean c

M7.3.j-2=3.the

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